Zero-Mode Dynamics of String Webs

Paul Shocklee[a,b,1] and Lárus Thorlacius[b,2]

a) Joseph Henry Laboratories
Princeton University
Princeton, New Jersey 08544
USA

b) University of Iceland
Science Institute
Dunhaga 3, 107 Reykjavik
Iceland

Abstract

At sufficiently low energy the dynamics of a string web is dominated by zero modes involving rigid motion of the internal strings. The dimension of the associated moduli space equals the maximal number of internal faces in the web. The generic web moduli space has boundaries and multiple branches, and for webs with three or more faces the geometry is curved. Webs can also be studied in a lift to M-theory, where a string web is replaced by a membrane wrapped on a holomorphic curve in spacetime. In this case the moduli space is complexified and admits a Kähler metric.

[1]shocklee@princeton.edu
[2]lth@hi.is

1   Introduction

A few years ago it was realized that type IIB string theory not only contains an $SL(2, \mathbf{Z})$ multiplet of (p, q) strings but also junctions where three or more of these strings come together [1, 2]. These junctions preserve a fraction of the IIB supersymmetry provided appropriate angles are chosen between the different (p, q) strings [3, 4]. A collection of supersymmetric string junctions can in turn be combined into a stable network of strings [4], also referred to as a string web. Such constructions are of intrinsic interest as solitonic solutions of IIB string theory but they have also found other applications. They, for example, play a key role in realizing exceptional gauge symmetry in type IIB theory [5]; they represent $\frac{1}{4}$-BPS states in D=4, $\mathcal{N} = 4$, $SU(N)$ supersymmetric Yang-Mills theory [6, 7, 8, 9, 10]; and they enter into the AdS/CFT determination of the potential between dyons at strong coupling in the $\mathcal{N} = 4$ gauge theory [11].

String webs are dynamical objects with a rich spectrum of excitations. Their dynamics has been studied in the small oscillation limit [12, 13] and for generic string webs a set of rigid zero modes has been identified [14, 15], which will dominate the dynamics at sufficiently low energies. In this paper we will primarily be interested in these zero-mode motions. We provide a general framework for studying the geometry of the associated moduli space, working out the explicit metric for some examples.[3]

Supersymmetric string junctions and string webs may also be represented in a lift to M-theory as a network of supermembranes holomorphically embedded into $\mathbf{R}^{1,8} \times \mathbf{T}^2$ where the integer charges of each (p, q) string are realized as winding numbers of the corresponding membrane segment around the cycles of the $\mathbf{T}^2$ [17, 18, 19]. The membrane representation allows a systematic derivation of the modular dynamics. The moduli space is complexified and has interesting geometric features which we explore alongside the more pedestrian string viewpoint.

The plan of the paper is as follows. Section 2 reviews basic facts about string junctions and string webs, and is followed by a discussion of the web moduli space and the dynamics of zero modes in Section 3. We explicitly work out the metric on moduli space for some simple examples and give arguments about boundaries and multiple branches in the general case. Section 4 describes the lift to M-theory and our concluding remarks are in Section 5. In an appendix we prove that the moduli space of any string web with only two internal faces is flat.

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[3]The low-energy dynamics of string lattices has been recently discussed in [16]. In the present paper, we are instead concerned with moduli of finite webs but the two systems are related.

In this paper, we treat only bosonic zero-modes. String webs are supersymmetric systems and when the fermionic zero modes are included the dynamics is governed by a supersymmetric effective theory which we intend to present in a forthcoming paper [20].

2 String junctions and string webs

Three (p, q) strings can come together to form a string junction, provided that the condition,

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} q_i = 0, \tag{1}$$

is satisfied, which is necessary for charge conservation. In order for this junction to be stable, the angles between the strings must be adjusted so that the net force on the vertex cancels. The tension of a (p, q) string is given by

$$T_{(p,q)} = T_{(1,0)} |p + q\tau|, \tag{2}$$

where $T_{(1,0)}$ is the fundamental string tension and τ is the axion-dilaton modulus of Type IIB theory. Stability is guaranteed if each (p, q) string is oriented along the vector $(p + q\tau)$ in the complex plane [3, 4]. In this paper, we will for simplicity consider the self-dual point, $\tau = i$, so that fundamental $(1, 0)$ strings are oriented horizontally in all the figures while Dirichlet $(0, 1)$ strings are vertical.

One then constructs string webs by joining some number of these vertices together. If all the strings in a given web lie in a single plane, with each string oriented parallel to its own charge vector, the web is stable and will in fact preserve $1/4$ of the IIB supersymmetry [4]. Of course, an overall rotation of the entire web does not affect its stability. The charge conservation and stability conditions at a vertex are invariant under a change of sign of all the charges. If we adopt the convention that strings are oriented outward from each vertex, there is no ambiguity in charge assignments for strings that form external legs of a web, but an internal string carries opposite charges with respect to each of the two vertices it ends at.

Certain supersymmetric string webs can be deformed without violating the local charge and stability conditions at any of the vertices. These are either global deformations that move the external strings as in Figure 1a, or local deformations that only move internal strings but leave the external ones unchanged as in Figure 1b. Note that n-point junctions with $n > 3$, as in Figure 1, are degenerate and can always be resolved into a web with only 3-point junctions by local and/or global deformations.

The total energy of a string web is preserved under local deformations. In other words, the total rest energy of the new strings that are created when a degenerate junction is "blown
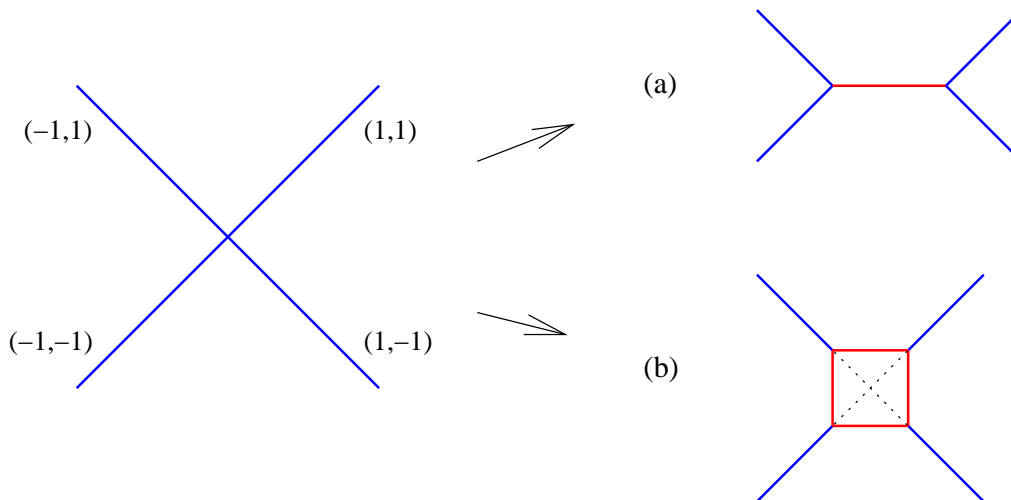
Figure 1: Web deformations: (a) global deformation, (b) local deformation.

up" as in Figure 1b precisely equals the total rest energy of the string segments that have been removed from the original web in the process (shown as dotted lines in the figure). Global deformations, on the other hand, involve shifting infinitely long segments of string and therefore do not correspond to physical motions of a string web but rather to changing its defining parameters. In other words, the worldsheet zero-modes that correspond to global deformations are not normalizable. In applications to D=4 supersymmetric gauge theory the external strings end on 3-branes, which extend in directions orthogonal to the plane of the web but do not move within that plane, so that in this case the web has no global deformations [6]. In the present paper we will focus on local deformations and the moduli that parametrize them.

3   Grid diagrams and zero-mode dynamics

The number of local moduli of a given string web is governed by the (p,q) charges carried by the external legs of the web and is equal to the maximal number of internal faces in the web. It can be read off from a "grid diagram", which is a dual description of the web [6, 14, 15]. The rules for drawing grid diagrams are as follows (Figures 2 and 3 provide examples):

1. The diagram is drawn on a two-dimensional, integer, square lattice.

2. A (p,q) string is represented by a $(-q,p)$ lattice vector.
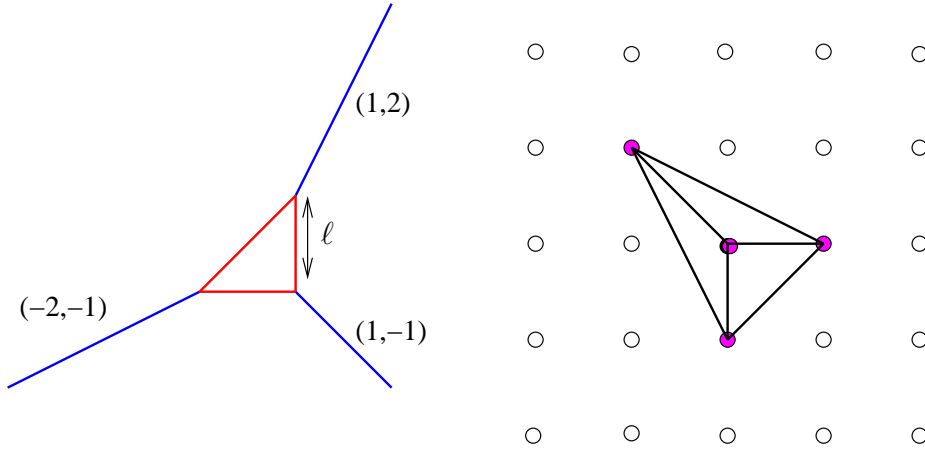
Figure 2: A web with a single zero mode and its grid diagram.

3. Starting at an arbitrary lattice point, the vectors representing the external strings are drawn in cyclic order, so that they form a convex polygon.

The number of local moduli is then given by the number of lattice points enclosed by the external polygon in the grid diagram. The grid diagram in Figure 2 has only one internal lattice point and thus the corresponding web has a single local modulus, which can, for example, be taken as the length ℓ of the vertical internal string. The web in Figure 3, on the other hand, has two local moduli.

The local moduli ℓ_i parametrize a set of web configurations that are degenerate in total rest energy of string. However, the total length of string involved in a rigid zero-mode motion on the moduli space clearly depends on the value of the moduli. In other words, the kinetic energy of zero-mode motion is a function of the location on moduli space, which in turn defines a dynamical metric on the moduli space in the usual manner,

$$S = \frac{1}{2} \int dt \, g_{ij}(\ell) \, \dot{\ell}_i \, \dot{\ell}_j \,. \tag{3}$$

As a simple example, let us consider the web in Figure 2. This web has only a single modulus which we can take to be the length ℓ of the vertical internal string. A change in ℓ leads to transverse motion of each internal string in addition to changing its length. The total mass of string that moves is proportional to ℓ with a coefficient that depends on the tension of the individual strings. Adding up the contribution from the three internal strings leads to the following effective zero-mode action,

$$S = \frac{T_{(1,0)}}{6} \int dt \, \ell \, \dot{\ell}^2 \,. \tag{4}$$
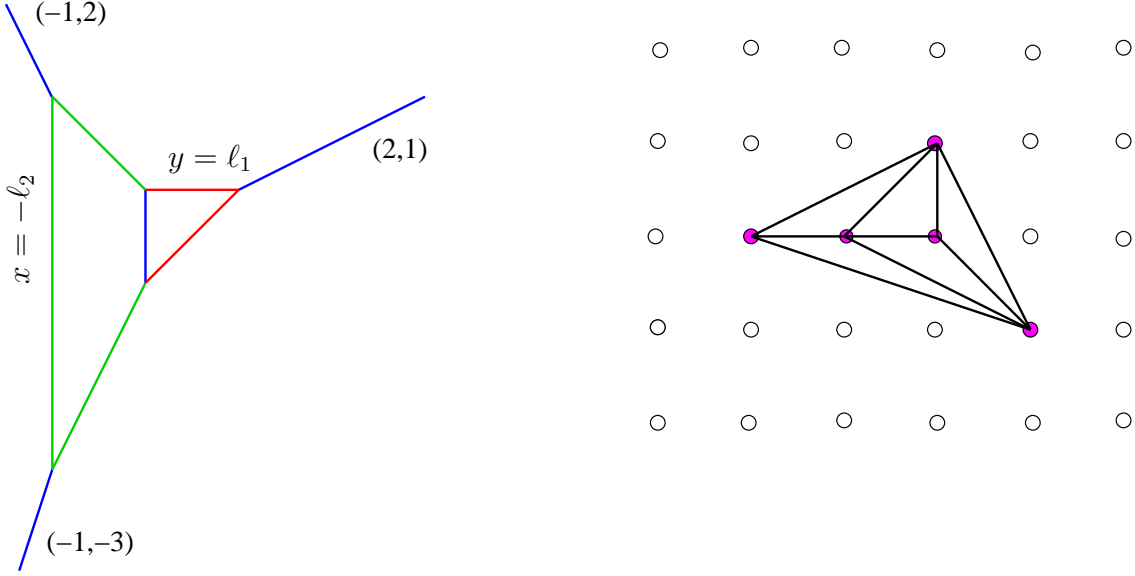
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Figure 3: A web with two zero modes and its grid diagram.

This is of course only the bosonic part. We'll discuss fermions and supersymmetry in [20]. The change of variables $u = \alpha \ell^{3/2}$, with $\alpha = 2\sqrt{T_{(1,0)}}/3\sqrt{3}$, takes the system to the standard one-dimensional free particle,

$$S = \frac{1}{2} \int dt \; \dot{u}^2 \,, \tag{5}$$

reflecting the fact that any one-dimensional metric is flat.

Now consider the more complicated web of Figure 3, where the moduli space is two-dimensional. We parametrize the moduli space by (ℓ_1, ℓ_2) as follows: Let the web lie in the xy-plane such that the external strings define the straight lines $y = \frac{1}{2}x$, $y = -2x$, and $y = 3x$. Then let the horizontal $(1,0)$ string be at $y = \ell_1$ and the leftmost vertical $(0,1)$ string at $x = -\ell_2$, as shown in Figure 3. This is enough to uniquely determine the web configuration and is a convenient parametrization to compare with the membrane picture which we develop below. Adding up contributions from the six internal strings one finds

$$S = \frac{T_{(1,0)}}{2} \int dt \; \left[3(3\ell_1 - \ell_2)\dot{\ell}_1^{\,2} - 2(3\ell_1 - \ell_2)\dot{\ell}_1\dot{\ell}_2 + (\ell_1 + 8\ell_2)\dot{\ell}_2^{\,2} \right] . \tag{6}$$

At first sight the moduli space metric appears non-trivial, but by defining $\tilde{\ell}_1 = \ell_1 - \frac{1}{3}\ell_2$, $\tilde{\ell}_2 = \ell_2$, followed by $\hat{\ell}_1 = \alpha_1 \tilde{\ell}_1^{\,3/2}$, $\hat{\ell}_2 = \alpha_2 \tilde{\ell}_2^{\,3/2}$, with appropriate constants α_1, α_2, we once again arrive at a manifestly flat metric. The geometry is flat but the moduli space has boundaries; in terms of our parameters, we have the constraints $0 \le \ell_1 \le 2\ell_2$ and $0 \le \ell_2 \le 3\ell_1$.
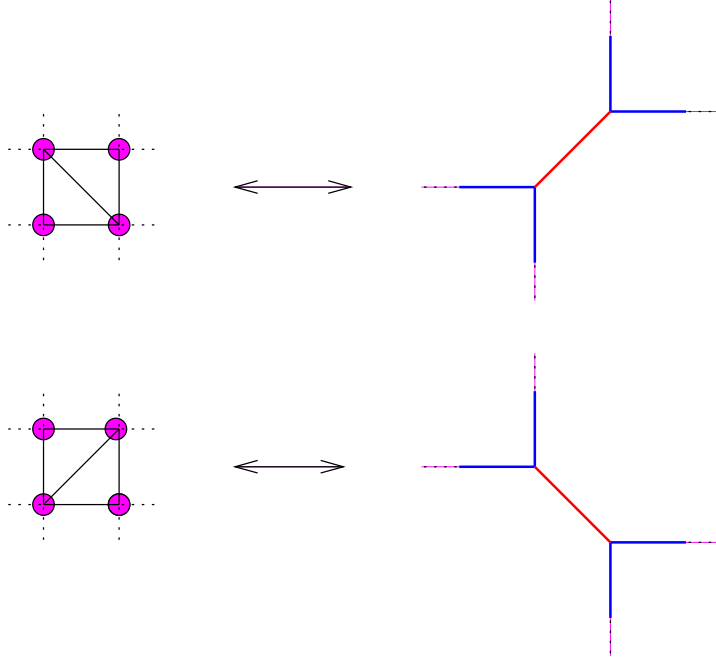
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Figure 4: Two branches in the moduli space related by a grid flip.

Grid diagrams with only two internal lattice points are restricted in form and this places strong constraints on the geometry. In fact, every two-parameter moduli space of string webs is flat. A proof of this statement is given in the appendix. By explicitly considering an example web with three internal faces we have, however, established that the metric generically has non-trivial curvature once the dimension of the moduli space is three or more. The flatness of the low-dimensional moduli spaces is presumably tied in with the high degree of supersymmetry of these systems.

A general feature of string web moduli spaces is that the metric coefficients can be written as linear functions of the moduli,

$$g_{ij} = \sum_{k=1}^{I} \alpha_{ijk} \, \ell_k \,, \tag{7}$$

where I is the maximal number of internal faces in the web and the α_{ijk} are functions of the (p, q) charges of the strings.

This method of adding "by hand" contributions to the kinetic energy from individual internal strings gets tedious for more complicated webs and one would like to find a more systematic approach. As the number of internal points in the grid diagram grows, the web moduli space not only becomes curved but it is also branched. To see this, consider

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a grid diagram where the convex polygon representing the external strings encloses a unit square in the grid lattice. If all the string junctions in the web are non-degenerate, *i.e.* connect only three strings, then this part of the grid diagram takes one of the two forms shown in Figure 4. The corresponding string configurations differ in the orientation of the string connecting two neighboring junctions as shown in the figure. When this structure is embedded in a larger string web there will in general be local deformations that take the system from one configuration to the other through a degenerate intermediate configuration containing a four-string junction. These degenerate webs correspond to a codimension one branching surface in the moduli space. For a web with many internal points in the grid diagram a large number of such "grid flips" is possible and the moduli space has multiple branches.

In the following section we describe how string webs may be represented as wrapped membranes in M-theory. In this approach the moduli space gets complexified, the branching is smoothed out, and we find a systematic prescription for computing the moduli space metric.

4   Lift to M-theory

In M-theory, string webs are represented by membranes wrapped on holomorphic curves embedded in $\mathbf{R}^{1,8} \times \mathbf{T}^2$ [17, 18]. Let us take the string web to lie in the $X^1 X^2$-plane and parametrize the torus by $X^9 \cong X^9 + 2\pi R$ and $X^{10} \cong X^{10} + 2\pi R$, where R is the compactification radius and we have set the torus modular parameter (which is to be identified with the type IIB modulus) to $\tau = i$. Now introduce complex coordinates

$$Z^1 = X^1 + iX^9, \qquad Z^2 = X^3 + iX^{10}, \tag{8}$$

and then change to global variables

$$u = \exp(Z^1/R), \qquad v = \exp(Z^2/R). \tag{9}$$

Consider a membrane embedded along the holomorphic curve defined by

$$u^q v^{-p} = \eta, \tag{10}$$

with η a non-zero complex constant and p, q integers. This curve represents a (p, q) string. Its projection onto the $X^1 X^2$-plane is

$$qX^1 - pX^2 = R \log |\eta|, \tag{11}$$

i.e. a straight line parallel to the (p, q) charge vector whose position is fixed by η. Meanwhile the projection onto the $\mathbf{T}^2$ shows that the membrane is wrapped on the (p, q) homology cycle.

A three-string junction consisting of a (p_1, q_1), (p_2, q_2), and a $(-p_1 - p_2, -q_1 - q_2)$ string is represented by a single membrane embedded along the curve

$$\eta_1 u^{q_1} v^{-p_1} + \eta_2 u^{-q_2} v^{p_2} = \eta_3 \,, \tag{12}$$

as is easily verified by considering the asymptotic behavior of the terms in (12) when X^1 and X^2 approach $\pm\infty$ in various combinations [17, 18]. The complex coefficients η_1, η_2 and η_3 serve to fix the position of the junction.

The holomorphic equation for an arbitrary string web can be written down directly from the grid diagram [14, 15]. Once an origin is chosen for the lattice, the relation is given by

$$\sum_{i}^{E} \eta_i u^{m_i} v^{n_i} + \sum_{j}^{I} \lambda_j u^{m_j} v^{n_j} = 0 \,, \tag{13}$$

where (m_i, n_i) are the lattice coordinates of the ith vertex. Shifting the lattice origin amounts to multiplying all the terms by a common factor and has no effect on the embedding. The coefficients η_i correspond to corner points of the convex polygon in the grid diagram while the λ_j correspond to internal points and hence internal faces of the web. Varying the η_i and λ_j amounts to global and local deformations of the web, respectively. In this paper we are primarily interested in local deformation so we focus our attention on the λ_j, which parametrize the membrane moduli space. The moduli space has I complex dimensions, where I is the number of internal faces of the web, or equivalently the genus of the embedded membrane in (13). The string description is recovered in the limit where the handles of the membrane become large compared to the compactification scale R, *i.e.* when $\log|\lambda_j| >> 1$.

At first sight, the membrane moduli space shows no indication of the boundaries and branches of the string web moduli space. For an infinitely extended web, each λ_j ranges over the entire complex plane.[4] It turns out, however, that the membrane moduli space has curvature singularities precisely in regions that correspond to boundaries of the string web moduli space. These singularities only occur when one or more of the moduli satisfy $\log|\lambda_j| \sim 1$, *i.e.* the size of a handle is shrinking to the compactification scale R. The locus of singularities is generally of (complex) codimension one in the membrane moduli space.

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[4]Webs with the external strings ending on three-branes are represented in M-theory by an open membrane with boundaries on five-branes that wrap the $\mathbf{T}^2$ and then the membrane moduli space has finite volume.

4.1 Zero-mode dynamics

Let us first confirm that the λ_j are moduli in the sense that the rest energy of the membrane is independent of the values they take. This amounts to the membrane area being unchanged as the λ_j are varied. Since the membrane is described by a holomorphic embedding it is a Kähler manifold of one complex dimension. The area is then given by the integral of the Kähler form over the membrane, and we must show that this is constant. In our complex structure, the Kähler form is given by the pullback of the following form to the membrane,

$$\omega = \frac{i}{2}(dZ^1 \wedge d\bar{Z}^1 + dZ^2 \wedge d\bar{Z}^2). \tag{14}$$

In terms of (u, v), this is

$$\omega = \frac{i}{2}R^2 \left\{ \frac{du \wedge d\bar{u}}{|u|^2} + \frac{dv \wedge d\bar{v}}{|v|^2} \right\}. \tag{15}$$

Integrating the pullback of this form over the membrane gives

$$A = \frac{R^2}{2} \int du d\bar{u} \left(\frac{1}{|u|^2} + \left| \frac{1}{v}\frac{\partial v}{\partial u} \right|^2 \right), \tag{16}$$

where we have chosen u and $\bar{u}$ as independent variables and v and $\bar{v}$ are expressed as functions of u and $\bar{u}$ through the embedding equation (13). This integral is divergent, but by introducing appropriate cutoffs one finds that the divergence is associated with the infinite membrane pieces that correspond to the infinite external strings [17].

Taking a partial derivative with respect to λ_i, and using the fact that, for a holomorphic embedding, $\bar{v}$ is neither a function of u nor λ_i, we get

$$\begin{aligned}
\frac{\partial A}{\partial \lambda_i} &= \frac{R^2}{2} \int du d\bar{u} \left(-\frac{1}{v^2}\frac{\partial v}{\partial \lambda_i}\frac{\partial v}{\partial u}\frac{1}{\bar{v}}\frac{\partial \bar{v}}{\partial \bar{u}} + \frac{1}{v}\frac{\partial^2 v}{\partial \lambda_i \partial u}\frac{1}{\bar{v}}\frac{\partial \bar{v}}{\partial \bar{u}} \right) \\
&= \frac{R^2}{2} \int du d\bar{u} \frac{\partial}{\partial u} \left(\frac{1}{v}\frac{\partial v}{\partial \lambda_i}\frac{1}{\bar{v}}\frac{\partial \bar{v}}{\partial \bar{u}} \right).
\end{aligned} \tag{17}$$

This could in principle get contributions from the asymptotic region $|u| \to \infty$ and from points where the expression inside the derivative diverges. As mentioned above, the only possible singularities would come from regions associated with the external strings and in those regions we have $\partial v/\partial \lambda_i \to 0$. This, in turn, follows from the fact that the asymptotic behavior of the membrane embedding in both u and v is governed by the η_i terms in (13). We conclude that the area is invariant under local deformations and the λ_j indeed are moduli.

Let us consider now the dynamics of the corresponding zero modes. In this paper, we are only considering bosonic fields so the dynamics is governed by the Nambu-Goto action

for the membrane,

$$S_{NG} = -T_{M2} \int d^3\sigma \, \sqrt{\det \gamma_{\alpha\beta}}, \tag{18}$$

where

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{19}$$

is the induced metric on the membrane worldvolume.

We'll use a static gauge

$$\sigma^0 = t, \qquad \sigma^1 = u, \qquad \sigma^2 = \bar{u}, \tag{20}$$

where $t \equiv X^0$ is embedding time, $u, \bar{u}$ are defined as before and $v, \bar{v}$ are obtained from the embedding equation (13). The membrane is embedded into the X^1, X^2, X^9, X^{10} directions only and we have $X^3 = \ldots X^8 = 0$ for the remaining coordinates in (19).

We now make the usual moduli space approximation, where we assume that no oscillation modes of the membrane are excited and that time dependence of the membrane embedding enters only through the moduli:

$$v = v(u; \lambda_j(t)). \tag{21}$$

Expanding the action for slowly varying moduli, we find

$$S_{NG} = T_{M2} \int dt \left(-A + \frac{1}{2} R^4 g_{i\bar{j}} \dot{\lambda}_i \dot{\bar{\lambda}}_j + \mathcal{O}(\dot{\lambda}^4) \right), \tag{22}$$

where A is the membrane area and the metric on moduli space is given by

$$g_{i\bar{j}} = \frac{1}{2} \int dud\bar{u} \, \frac{1}{|uv|^2} \frac{\partial v}{\partial \lambda_i} \frac{\partial \bar{v}}{\partial \bar{\lambda}_j}. \tag{23}$$

This is manifestly a Kähler metric, $g_{i\bar{j}} = \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \bar{\lambda}_j} \mathcal{K}$, with the Kähler potential reducing to a particularly simple formal expression in terms of $Z^2(Z^1)$,

$$\mathcal{K} = \frac{1}{2R^4} \int dZ^1 d\bar{Z}^1 |Z^2|^2. \tag{24}$$

4.2 Explicit examples

Let us now apply this formalism to concrete examples. This reveals some of the geometry involved and allows us to test the general formula (23) by checking explicit answers against the corresponding string web results.

Consider the junction in Figure 2. It has one local deformation, and is described by the curve

$$u + u^2 v + v^2 + \lambda uv = 0. \tag{25}$$

Using equation (23), the metric on this moduli space is given by

$$g_{\lambda\bar\lambda} = \frac{1}{2}\int du d\bar u \; \frac{1}{|u^2+2v+\lambda u|^2}. \tag{26}$$

Since the curve (25) is quadratic in v, we can solve for v, leaving us with the following explicit integral expression for the metric,

$$g_{\lambda\bar\lambda} = \frac{1}{2}\int du d\bar u \frac{1}{|u(u(u+\lambda)^2-4)|}. \tag{27}$$

To make contact with the string web we consider the limit of large $|\lambda|$. In this limit, the internal face of the junction becomes large compared to the M-theory compactification scale R. By examining the projection of the curve (25) onto the X^1X^2-plane one identifies the string web modulus as $\ell = 3R\log|\lambda|$. We then write the complex membrane modulus as

$$\lambda = \exp(\frac{l}{3R}+i\theta), \tag{28}$$

and consider the change of variables from $(\lambda,\bar\lambda)$ to (l,θ). In the limit $\ell >> R$ one finds that the $g_{\theta\theta}$ term in the membrane kinetic energy may be dropped in comparison to the $g_{\ell\ell}$ term, effectively reducing the dimension of the moduli space from one complex to one real dimension. Furthermore, a numerical evaluation of the integral (27) shows that the kinetic part of the membrane action (22) reduces to the string web action (4), including the correct normalization, when we identify the fundamental string tension as $T_{(1,0)} = 2\pi R T_{M2}$.

The integrand in (27) is singular whenever the denominator has a zero. These singularities are integrable, unless there is a double zero, and then the metric itself is singular. In our example, this occurs when $\lambda = -3$, $3e^{\pi i/3}$, $3e^{-\pi i/3}$, and one easily checks that these are curvature singularities. They all occur at a distance of order R away from the origin in the X^1X^2-plane, which corresponds to the $\ell = 0$ boundary of the string web moduli space, in the $\ell >> R$ string limit.

Let us also consider a two-dimensional example corresponding to the string web in Figure 3. The algebraic curve for this junction has two deformation parameters, which we can call λ_1 and λ_2. The curve is given by the equation

$$u^2v^2 - v - u^3 + \lambda_1 u^2 v + \lambda_2 uv = 0. \tag{29}$$

The membrane parameters are related to those in the string picture by $\lambda_i = \exp(\ell_i/R)$. This curve is also quadratic in v, so we can again solve for v and express the metric on the moduli space explicitly in terms of integrals,

$$g_{i\bar j}\dot\lambda_i\dot{\bar\lambda}_j = \frac{1}{2}\int du d\bar u \frac{1}{|(1-\lambda_1 u^2 - \lambda_2 u)^2 + 4u^5|}(|u|^2|\dot\lambda_1|^2 + |\dot\lambda_2|^2 + u\dot\lambda_1\dot{\bar\lambda}_2 + \bar u\dot{\bar\lambda}_1\dot\lambda_2). \tag{30}$$

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In this case, the moduli space has two complex dimensions, and it has singularities that lie on complex lines, corresponding to the boundaries of the moduli space in the string limit. We have evaluated these integrals numerically in the limit of large $|\lambda_i|$, and we find that they reproduce the kinetic energy (6) of the string web moduli. It is interesting to note that the two-dimensional moduli space of the string web is everywhere flat, but may nevertheless be obtained as a slice through a non-trivial geometry, of complex dimension two, that has curvature singularities.

5 Conclusions

We have presented two methods for computing the metric on moduli space for a string web. One is to add up the kinetic energies of all the strings that are involved in the zero-mode motion. This method provides some physical insight but the necessary plane geometry gets tedious as the number of strings in the web grows. The moduli space of a string web with one or two internal faces is flat but has boundaries. The higher-dimensional moduli spaces have boundaries and also non-trivial curvature and topology, with multiple branches.

The second method involves lifting to M-theory where the string web is represented as a membrane holomorphically embedded in $\mathbf{R}^{1,8} \times \mathbf{T}^2$. The resulting moduli space is complex and admits a Kähler metric, which is directly obtained, up to quadrature, from the analytic curve defining the membrane embedding. Numerical evaluation of the required integrals in the appropriate limit gives results consistent with the previous method. One can obtain useful information by analytic methods without explicitly evaluating the metric. Although the complex moduli space has no boundaries, one instead finds curvature singularities in regions where the corresponding string web has internal faces shrinking to zero size. We expect regions of strong curvature to be repulsive in the effective quantum dynamics on the moduli space and thus to mimick the effect of boundaries. The actual singularities should perhaps not be taken too literally since they occur in regions of moduli space where the size of a membrane handle approaches the M-theory compactification scale R, and the classical supergravity description of our membranes is breaking down. One expects additional light degrees of freedom to emerge in the theory at that point, which resolve the singularity. Such degrees of freedom are readily apparent in the string web picture, where fundamental strings, attached to the solitonic strings of the web and stretched across a web face, become massless when the face shrinks to zero size, and need to be included in the low-energy dynamics.

Our description of the zero-mode dynamics of superstring webs in this paper is missing some crucial ingredients. In particular, we need to incorporate fermion zero modes in order
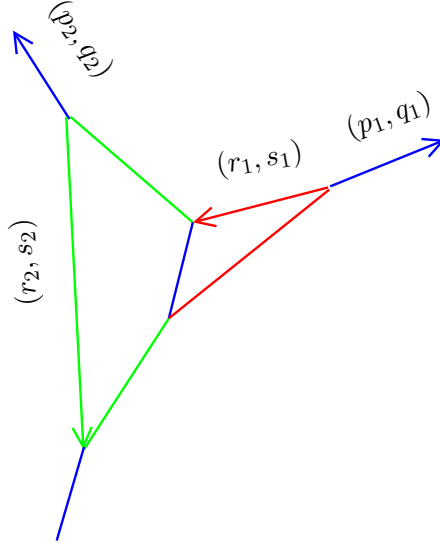
Figure 5: A generic string web with two zero modes.

to explore the supersymmetry of the low-energy dynamics. Our methods can be generalized to include fermions by considering supermembranes embedded in the superspace of $D = 11$ supergravity [20].

Appendix

A generic string web with three external strings and two internal faces is shown in Figure 5. It is a general feature of such webs that one of the faces meets only one of the external strings, to which we have assigned the charges (p_1, q_1), while the other face connects to the two remaining external strings. By an overall rotation we can always make the (p_1, q_1) string lie in the upper-right-hand quadrant as in the figure. We have indicated the charges and orientation of a few of the strings in the web. The charge assignments for the remaining strings follow uniquely from charge conservation at the vertices.

We have, for convenience, arranged the global moduli to be such that when both faces shrink to zero size, the three external strings meet at a point, which we take as our origin in the xy-plane of the web. The external strings then lie on the straight lines $p_1 y = q_1 x$,

$p_2 y = q_2 x$, and $(p_1 + p_2)y = (q_1 + q_2)x$. We introduce the two moduli ℓ_1, ℓ_2 by parametrizing the two labelled internal strings in Figure 5 as follows: $r_1 y = s_1 x - \ell_1$ and $r_2 y = s_2 x - \ell_2$. Note that the web in Figure 3 is a special case of this construction with $p_1 = 2$, $q_1 = 1$, $p_2 = -1$, $q_2 = 2$, $r_1 = -1$, $s_1 = 0$, $r_2 = 0$, and $s_2 = -1$.

It is now a straightforward, if tedious, exercise to add up the kinetic energies of the six internal strings when ℓ_1 and ℓ_2 depend slowly on time. The result is a rather messy expression constructed out of $SL(2, Z)$ invariants of the form $p_i q_j - p_j q_i$ involving various combinations of the string charges. Rather than presenting all the metric components here and establishing flatness by direct computation, we use the following shortcut. As mentioned in the main text, generic metric components are linear functions of the moduli,

$$g_{ij} = a_{ij}\ell_1 + b_{ij}\ell_2, \tag{31}$$

with $i, j = 1, 2$. The coefficients a_{ij}, b_{ij} are functions of $SL(2, Z)$ invariants. This metric is certainly flat if it can be brought into the form

$$[g_{\tilde{i}\tilde{j}}] = \begin{bmatrix} \alpha_1 \tilde{\ell}_1 & 0 \\ 0 & \alpha_2 \tilde{\ell}_2 \end{bmatrix}, \tag{32}$$

by a linear transformation of the moduli,

$$\begin{pmatrix} \tilde{\ell}_1 \\ \tilde{\ell}_2 \end{pmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}. \tag{33}$$

The metric coefficients satisfy the following relations:

$$a_{11} = A^3 \alpha_1 + C^3 \alpha_2, \qquad b_{11} = A^2 B \alpha_1 + C^2 D \alpha_2, \tag{34}$$
$$a_{12} = A^2 B \alpha_1 + C^2 D \alpha_2, \quad b_{12} = AB^2 \alpha_1 + CD^2 \alpha_2, \tag{35}$$
$$a_{11} = AB^2 \alpha_1 + CD^2 \alpha_2, \quad b_{11} = B^3 \alpha_1 + D^3 \alpha_2. \tag{36}$$

We note from this that a sufficient condition for flatness is to have the relations $b_{11} = a_{12}$ and $b_{12} = a_{22}$ satisfied by our original metric, while a_{11} and b_{22} are unrestricted. By explicit calculation we find

$$b_{11} = a_{12} = (q_1(p_2 + r_2) - p_1(q_2 + s_2))/D, \tag{37}$$
$$b_{12} = a_{22} = (s_1 p_1 - r_1 q_1)/D, \tag{38}$$

with $D = (s_1(p_2 + r_2) - r_1(q_2 + s_2))((p_1 + r_1)(q_1 + q_2 + s_2) - (q_1 + s_1)(p_1 + p_2 + r_2))$, showing that the moduli space of a generic web with two internal faces is indeed flat.

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